ON TRANSFORMATION OF KINETIC ENERGY BETWEEN THE VERTICAL SHEAR FLOW AND THE VERTICAL MEAN FLOW IN THE ATMOSPHERE

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[Manuscript received February 15, 1962; revised May 21, 1962]

ABSTRACT

The kinetic energy of the horizontal, hydrostatic flow is divided into the kinetic energies of the vertically integrated flow and the deviation from this flow, the so-called shear flow. The energy transformation between the two types of flow is found in the general case of the primitive equations and also for the most simple quasi-non-divergent model. The two transformations are discussed, and the energy transformation in the quasi-non-divergent model in the two-parameter case is discussed as a function of wave number using linear theory. The energy conversion has been computed on a daily basis for the month of January 1959, and compared with earlier results of computations of transformations between available potential energy and shear flow kinetic energy. It is shown that the latter conversion changes the kinetic energy of the shear flow and not that of the mean flow. The residence time is estimated for the shear flow as well as the mean flow.

The energy transformation between the vertical shear flow and mean flow due to the non-divergent and divergent flow has been computed in the wave-number regime for the first 10 zonal wave numbers for each day in January 1959. It is found that the energy conversion between shear flow and mean flow is about 30 percent of the conversion between the available potential energy and the shear flow kinetic energy.

A further result is that the energy conversion between the shear flow and the mean flow due to the divergent part of the flow is estimated to be negative and about 10 percent of the conversion due to the non-divergent part of the flow.

The energy conversion as a function of wave number shows a maximum for the most unstable baroclinic waves.

1. INTRODUCTION

During recent years the energetics of the atmosphere have received much attention in studies of the general circulation. The energy conversions which take place between the different forms of energy have been computed in theoretical studies (Phillips [5], Charney [2]), in numerical experiments (Phillips [6]) and in observational studies (Wiin-Nielsen [10], Saltzman and Fleischer [9]). Most of the work has been done in the evaluation of the conversion between potential and kinetic energy. This conversion has been computed from the latitudinal average of the flow, for the deviations from the averaged flow, and as a function of the zonal wave number.

In the present study we shall again consider the energy conversion between potential and kinetic energy, but we shall divide the kinetic energy of the flow into two parts: the kinetic energy of the vertically integrated flow and the kinetic energy of the deviation from this flow, which in the present study will be called the shear flow.

The total energy conversion between potential and kinetic energy computed from observations is found to be positive at any time. One might think that the kinetic energy created by conversion from potential energy could be used partly to increase the kinetic energy of the vertically averaged flow and partly to increase the shear-flow kinetic energy. The energy conversion between potential and kinetic energy depends on the correlation between the vertical velocity and the temperature. Excluding external gravity waves by a simplified lower boundary condition, it is well-known that the vertically averaged flow becomes non-divergent. It is therefore to be expected that the kinetic energy created by conversion from potential energy will increase the kinetic energy of the shear flow.

The investigation will proceed along the following lines: We shall first show that the kinetic energy of the horizontal, hydrostatic flow can be expressed as the sum of the kinetic energy of the vertically averaged flow and the shear flow. Next, we shall show that energy converted from potential energy goes into the shear flow. It follows then that there must be a transformation of energy between the shear flow and the vertically averaged flow. When we have determined this energy transformation function, which we shall speak about as transformation between shear flow and mean flow, we are in a position to determine the mechanism which controls whether the kinetic energy is stored in the shear flow or in the vertical mean flow.

The energy conversion between the shear flow and the mean flow is first determined in the general case of the non-filtered equations. Next, we find the same energy conversion for the filtered (quasi-non-divergent) equation, and we can make a comparison between the two conversions.

In order to get an insight into how the energy conversion may depend upon the scale of the motion we finally use, as an example, simple sinusoidal two-dimensional waves to compute the energy conversion as a function of the wavelength.

2. KINETIC ENERGY OF MEAN FLOW AND SHEAR FLOW

The vertically integrated flow will be defined by the following operator

$$\overline{(\ )}=\frac{1}{p_0}\int_0^{p_0}(\ )dp \tag{2.1}$$

where p is pressure and p_0 the surface pressure.

Using (2.1) we may write the components of the horizontal wind in the form

$$u=\overline{u}+u', \qquad v=\overline{v}+v' \tag{2.2}$$

where naturally

$$\overline{u'}=\overline{v'}=0 \tag{2.3}$$

The total kinetic energy will be defined by the integral

$$K=\int_0^\infty\int_S\frac{1}{2}\rho(u^2+v^2)dSdz=\frac{1}{g}\int_0^{p_0}\int_S\frac{1}{2}(u^2+v^2)dSdp \tag{2.4}$$

where ρ is the density, S the region of the whole sphere, and where we have made use of the hydrostatic equation to obtain the last integral in (2.4).

Introducing the relations (2.2) in (2.4) and making use of the relations (2.3) we obtain

$$K=\frac{1}{g}\int_0^{p_0}\int_S\left[\frac{1}{2}(\overline{u}^2+\overline{v}^2)+\frac{1}{2}(u'^2+v'^2)\right]dSdp \tag{2.5}$$

or

$$K=\overline{K}+K' \tag{2.6}$$

where

$$\left.\begin{aligned}K&=\frac{1}{g}\int_0^{p_0}\int_S kdSdp, & k&=\frac{1}{2}(u^2+v^2)\\ \overline{K}&=\frac{p_0}{g}\int_S\overline{k}dS, & \overline{k}&=\frac{1}{2}(\overline{u}^2+\overline{v}^2)\\ K'&=\frac{1}{g}\int_0^{p_0}\int_S k'dSdp, & k'&=\frac{1}{2}(u'^2+v'^2)\end{aligned}\right\} \tag{2.7}$$

K will be called the total kinetic energy, $\overline{K}$ the kinetic energy of the vertical mean flow, and K' the kinetic energy of the shear flow. It will be noticed that we have not included the vertical motion in the evaluations of the energies.

The use of the hydrostatic equation filters out sound waves, and we shall, for simplicity, in the following also exclude external gravity waves by using the boundary conditions

$$\omega=\frac{dp}{dt}=0, p=0, p=p_0=100\text{ cb.} \tag{2.8}$$

3. ENERGY CHANGES IN TOTAL FLOW, MEAN FLOW, AND SHEAR FLOW

The equations of motion and the continuity equation will be used in the following form:

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\omega\frac{\partial u}{\partial p}=-\frac{\partial\phi}{\partial x}+fv+F_x\\ &\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\omega\frac{\partial v}{\partial p}=-\frac{\partial\phi}{\partial y}-fu+F_y\\ &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial\omega}{\partial p}=0\end{aligned}\right\} \tag{3.1}$$

In (3.1) $\phi=gz$ is the geopotential, f the Coriolis parameter, g the acceleration of gravity, F_x and F_y the two components of the frictional force per unit mass.

Multiplying the first equation of motion by u, the second by v, adding the two resulting equations, and then integrating over the complete atmosphere, it has been shown earlier (see for example Wiin-Nielsen [10]) that

$$\frac{dK}{dt}=\int_0^{p_0}\int_S\omega\frac{\partial z}{\partial p}dSdp+\int_0^{p_0}\int_S\mathbf{V}\cdot\mathbf{F}dSdp \tag{3.2}$$

Our next object is to derive an equation for the rate of change of the kinetic energy of the mean flow, $d\overline{K}/dt$. In order to do this it is necessary first to obtain the equations of motion of the vertically averaged flow. These equations are derived by introducing (2.2) in the system (3.1) and applying the operator (2.1). We arrive in this way at the following set of equations:

$$\left.\begin{aligned}&\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{\left[u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}+\omega\frac{\partial u'}{\partial p}\right]}\\ &\qquad\qquad=-\frac{\partial\overline{\phi}}{\partial x}+f\overline{v}+\overline{F}_x\\ &\frac{\partial\overline{v}}{\partial t}+\overline{u}\frac{\partial\overline{v}}{\partial x}+\overline{v}\frac{\partial\overline{v}}{\partial y}+\overline{\left[u'\frac{\partial v'}{\partial x}+v'\frac{\partial v'}{\partial y}+\omega\frac{\partial v'}{\partial p}\right]}\\ &\qquad\qquad=-\frac{\partial\overline{\phi}}{\partial y}-f\overline{u}+\overline{F}_y\\ &\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{v}}{\partial y}=0\end{aligned}\right\} \tag{3.3}$$

The system of equations (3.3) governs the development of the vertically integrated flow. The last terms in the brackets on the left sides measure the contributions from the shear flow to the local accelerations in the mean flow. The vertically integrated continuity equation says that the vertical mean flow is non-divergent, i.e., $\nabla\cdot\overline{\mathbf{V}}=0$. The latter property is due to our simplified lower boundary condition $\omega=0$ for $p=p_0$.

We obtain now the rate of change of the kinetic energy of the mean flow by multiplying the first equation in (3.3) by $\overline{u}$, the second by $\overline{v}$, adding the two resulting equations and integrating over the complete atmosphere, making use of the third equation in the system (3.3). When this procedure is carried out, we arrive at the following equation:

$$\frac{d\overline{K}}{dt}=-\frac{p_0}{g}\int_S\left[\overline{u}\cdot\overline{\left\{u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}+\omega\frac{\partial u'}{\partial p}\right\}}+\overline{v}\,\overline{\left\{u'\frac{\partial v'}{\partial x}+v'\frac{\partial v'}{\partial y}+\omega\frac{\partial v'}{\partial p}\right\}}\right]dS+\frac{p_0}{g}\int_S\overline{\mathbf{V}}\cdot\overline{\mathbf{F}}dS \quad (3.4)$$

We may transform the integrand in the first integral of (3.4) first in the following way:

$$\overline{\left\{u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}+\omega\frac{\partial u'}{\partial p}\right\}}=\overline{\frac{\partial u'u'}{\partial x}+\frac{\partial u'v'}{\partial y}} \quad (3.5)$$

$$\overline{\left\{u'\frac{\partial v'}{\partial x}+v'\frac{\partial v'}{\partial y}+\omega\frac{\partial v'}{\partial p}\right\}}=\overline{\frac{\partial u'v'}{\partial x}+\frac{\partial v'v'}{\partial y}} \quad (3.6)$$

The last two expressions may further be transformed using the identities

$$\frac{\partial u'u'}{\partial x}+\frac{\partial u'v'}{\partial y}=\frac{\partial}{\partial x}\left[\frac{1}{2}(u'^2+v'^2)\right]+u'\nabla\cdot\mathbf{V}'-v'\zeta' \quad (3.7)$$

$$\frac{\partial u'v'}{\partial x}+\frac{\partial v'v'}{\partial y}=\frac{\partial}{\partial y}\left[\frac{1}{2}(u'^2+v'^2)\right]+v'\nabla\cdot\mathbf{V}'+u'\zeta' \quad (3.8)$$

where ζ' is the relative vorticity of the shear flow.

Using the relations (3.5)–(3.8) we may write (3.4) in the form

$$\frac{d\overline{K}}{dt}=-\frac{p_0}{g}\int_S[\overline{\mathbf{V}}\cdot\nabla k'+\overline{\mathbf{V}}\cdot\overline{(\nabla\cdot\mathbf{V}')\mathbf{V}'}+(\overline{\mathbf{V}}\times\mathbf{k})\cdot\overline{\zeta'\mathbf{V}'}]dS+\frac{p_0}{g}\int_S\overline{\mathbf{V}}\cdot\overline{\mathbf{F}}dS \quad (3.9)$$

or

$$\frac{d\overline{K}}{dt}=-\frac{p_0}{g}\int_S[\overline{\mathbf{V}}\cdot\overline{(\nabla\cdot\mathbf{V}')\mathbf{V}'}+(\overline{\mathbf{V}}\times\mathbf{k})\cdot\overline{\zeta'\mathbf{V}'}]dS+\frac{p_0}{g}\int_S\overline{\mathbf{V}}\cdot\overline{\mathbf{F}}dS \quad (3.10)$$

If we subtract (3.10) from (3.2) we get:

$$\frac{dK'}{dt}=\frac{p_0}{g}\int_S[\overline{\mathbf{V}}\cdot\overline{(\nabla\cdot\mathbf{V}')\mathbf{V}'}+(\overline{\mathbf{V}}\times\mathbf{k})\cdot\overline{\zeta'\mathbf{V}'}]dS+\int_0^{p_0}\int_S\omega\frac{\partial z}{\partial p}dSdp+\frac{1}{g}\int_0^{p_0}\int_S\mathbf{V}'\cdot\mathbf{F}'dSdp \quad (3.11)$$

While (3.2) gives the change of the total kinetic energy, (3.10) and (3.11) determine the rate of change of the kinetic energies of the mean flow and the shear flow. On the basis of these formulas we may state that the last integral in (3.10) measures the frictional dissipation of the kinetic energy of the mean flow. This integral depends only on the mean wind and the mean frictional force. The last integral of (3.11) gives the frictional dissipation of the shear flow. This integral contains only the shear wind and the deviation of the frictional force from its mean value.

With respect to the direct conversion of potential to kinetic energy, measured by the first integral in (3.2), we notice that this integral only appears in (3.11). This means that the kinetic energy, created by conversion from potential energy, goes directly into the reservoir of the kinetic energy of the shear flow.

Finally, the first integral appearing in (3.10), and with the opposite sign in (3.11), measures the energy conversion between the shear flow and the mean flow. Since we are going to investigate this integral in some detail in the following sections, we shall denote it

$$\{K'\cdot\overline{K}\}=-\frac{p_0}{g}\int_S[\overline{\mathbf{V}}\cdot\overline{(\nabla\cdot\mathbf{V}')\mathbf{V}'}+(\overline{\mathbf{V}}\times\mathbf{k})\cdot\overline{\zeta'\mathbf{V}'}]dS \quad (3.12)$$

If the integral is positive, we have a conversion from the kinetic energy of the shear flow to the kinetic energy of the mean flow.

In the general form (3.12) $\{K'\cdot\overline{K}\}$ depends on the value of two integrals. The integrand in each integral is a scalar product of two vectors. In the first integrand we find the scalar product of the mean wind, $\overline{\mathbf{V}}$, and the vertical average of the shear wind weighted with the divergence. The second integrand is the scalar product of the mean wind turned 90 degrees in a clockwise direction and the vertical average of the shear wind weighted with a relative vorticity of the shear flow.

The last term in (3.10), which is the opposite of the frictional dissipation of the mean flow, is most likely negative since the mean frictional force tends to be opposite to the mean wind. In the long term average it follows therefore that $\{K'\cdot\overline{K}\}$ measured by (3.12) must be positive since the kinetic energy of the mean flow probably does not change significantly in the mean over a long time.

A further discussion of the relative importance and interpretation of the two terms in (3.12) will be given in the later sections, but we notice that a numerical evaluation of both of the terms is possible from atmospheric wind data supplemented by a diagnostic computation of the horizontal divergence.

4. ENERGY TRANSFORMATIONS BETWEEN SHEAR FLOW AND MEAN FLOW IN QUASI-NON-DIVERGENT MODELS

The derivation in the preceding section was based on the non-filtered equations of motion. It is of interest to find the energy conversion between shear flow and mean flow also in the filtered equation or in other words, in a quasi-non-divergent model. It is to be expected that the first integral in (3.12) will be missing in this model, since it appears due to the divergence of the horizontal, isobaric wind. This divergence is neglected in the most simple quasi-non-divergent model. We shall in the derivation use the mean wind and the shear wind as defined by (2.2) and (2.3) and also the kenetic energies as given by (2.7) except that the horizontal wind components, u and v, now are considered to be non-divergent.

The prognostic equation for the model is now the vorticity equation in the form:

$$\frac{\partial\zeta}{\partial t}+\mathbf{V}_\psi\cdot\nabla(\zeta+f)=f_0\frac{\partial\omega}{\partial p}+\mathbf{k}\cdot(\nabla\times\mathbf{F}) \qquad (4.1)$$

From (2.2) it follows that we may write the wind and the vorticity in the forms

$$\mathbf{V}=\overline{\mathbf{V}}+\mathbf{V}', \qquad \zeta=\overline{\zeta}+\zeta' \qquad (4.2)$$

Substituting (4.2) and (4.1) and then applying the operator (2.1) and the boundary condition (2.8) we get:

$$\frac{\partial\overline{\zeta}}{\partial t}+\overline{\mathbf{V}}\cdot\nabla(\overline{\zeta}+f)+\overline{\mathbf{V}'\cdot\nabla\zeta'}=\mathbf{k}\cdot(\nabla\times\overline{\mathbf{F}}) \qquad (4.3)$$

If we subtract (4.3) from (4.1) after substitution of (4.2), we obtain the prognostic equation for the shear flow:

$$\frac{\partial\zeta'}{\partial t}+\overline{\mathbf{V}}\cdot\nabla\zeta'+\mathbf{V}'\cdot\nabla(\overline{\zeta}+f)+\mathbf{V}'\cdot\nabla\zeta'-\overline{\mathbf{V}'\cdot\nabla\zeta'}=f_0\frac{\partial\omega}{\partial p}+\mathbf{k}\cdot(\nabla\times\mathbf{F}') \qquad (4.4)$$

The kinetic energy of the mean flow may in this case be written

$$\overline{K}=\frac{p_0}{g}\int_S\frac{1}{2}\nabla\overline{\psi}\cdot\nabla\overline{\psi}dS \qquad (4.5)$$

since we only consider the kinetic energy of the non-divergent wind. From (4.5) it follows that:

$$\frac{d\overline{K}}{dt}=\frac{p_0}{g}\int_S\nabla\overline{\psi}\cdot\nabla\frac{\partial\overline{\psi}}{\partial t}dS=-\frac{p_0}{g}\int_S\overline{\psi}\frac{\partial\overline{\zeta}}{\partial t}dS \qquad (4.6)$$

In a similar way we obtain

$$K'=\frac{1}{g}\int_0^{p_0}\int_S\frac{1}{2}\nabla\psi'\cdot\nabla\psi'dSdp \qquad (4.7)$$

from which it follows that

$$\frac{dK'}{dt}=-\frac{1}{g}\int_0^{p_0}\int_S\psi'\frac{\partial\zeta'}{\partial t}dSdp \qquad (4.8)$$

It is seen from (4.6) and (4.8) that we can obtain expressions for $d\overline{K}/dt$ and dK'/dt by multiplying (4.3) by $\overline{\psi}$, (4.4) by ψ', and then performing the integrations. Applying this procedure first to (4.3) we get:

$$\overline{\psi}\frac{\partial\overline{\zeta}}{\partial t}+\overline{\psi}\,\overline{\mathbf{V}}\cdot\nabla(\overline{\zeta}+f)+\overline{\psi}\,\overline{\mathbf{V}'\cdot\nabla\zeta'}=\overline{\psi}\mathbf{k}\cdot(\nabla\times\overline{\mathbf{F}}) \qquad (4.9)$$

The second term will integrate to zero, because

$$\int_S\overline{\psi}\,\overline{\mathbf{V}}\cdot\nabla(\overline{\zeta}+f)dS=\int_S\nabla\cdot[\overline{\psi}(\overline{\zeta}+f)\overline{\mathbf{V}}]dS=0 \qquad (4.10)$$

while the third term in (4.9) will be integrated to

$$\int_S\overline{\psi}\,\overline{\mathbf{V}'\cdot\nabla\zeta'}dS=-\int_S\overline{\zeta'\mathbf{V}'}\cdot\nabla\overline{\psi}dS \qquad (4.11)$$

and therefore:

$$\frac{d\overline{K}}{dt}=-\frac{p_0}{g}\int_S\overline{\zeta'\mathbf{V}'}\cdot\nabla\overline{\psi}dS+\frac{p_0}{g}\int_S\left(-\frac{\partial\overline{\psi}}{\partial y}F_x+\frac{\partial\psi}{\partial x}F_y\right)dS \qquad (4.12)$$

but since $\nabla\overline{\psi}=\overline{\mathbf{V}}\times\mathbf{k}$, we have:

$$\frac{d\overline{K}}{dt}=-\frac{p_0}{g}\int_S[\overline{\mathbf{V}}\times\mathbf{k}]\cdot\overline{\zeta'\mathbf{V}'}dS+\frac{p_0}{g}\int_S\overline{\mathbf{V}}\cdot\overline{\mathbf{F}}dS \qquad (4.13)$$

Equation (4.13) should be compared with the first integral in (3.10), and it is seen, as expected, that the first part of the integral is missing in the quasi-geostrophic formulation.

Multiplying (4.4) by ψ' we obtain

$$\psi'\frac{\partial\zeta'}{\partial t}+\psi'\overline{\mathbf{V}}\cdot\nabla\zeta'+\psi'\mathbf{V}'\cdot\nabla(\overline{\zeta}+f)+\psi'\mathbf{V}'\cdot\nabla\zeta'-\psi'\overline{\mathbf{V}'\cdot\nabla\zeta'}=\psi'f_0\frac{\partial\omega}{\partial p}+\psi'\mathbf{k}\cdot(\nabla\times\mathbf{F}') \qquad (4.14)$$

Taking these terms one by one we obtain:

$$\int_0^{p_0}\int_S\psi'\overline{\mathbf{V}}\cdot\nabla\zeta'dSdp=p_0\int_S\overline{\zeta'\mathbf{V}'}\cdot\nabla\overline{\psi}dS \qquad (4.15)$$

$$\int_0^{p_0}\int_S\psi'\mathbf{V}'\cdot\nabla(\overline{\zeta}+f)dSdp=0 \qquad (4.16)$$

$$\int_0^{p_0}\int_S\psi'\mathbf{V}'\cdot\nabla\zeta'dSdp=0 \qquad (4.17)$$

$$\int_0^{p_0}\int_S\psi'\overline{\mathbf{V}'\cdot\nabla\zeta'}dSdp=0 \qquad (4.18)$$

Consequently:

$$\frac{dK'}{dt}=\frac{p_0}{g}\int_S\overline{\zeta'\mathbf{V}'}\cdot\nabla\overline{\psi}dS+\int_0^{p_0}\int_S\omega\frac{\partial z'}{\partial p}dSdp+\frac{1}{g}\int_0^{p_0}\int_S\mathbf{V}'\cdot\mathbf{F}'dSdp \qquad (4.19)$$

or:

$$\frac{dK'}{dt}=\frac{p_0}{g}\int_S\overline{\mathbf{V}}\times\mathbf{k}\cdot\overline{\zeta'\mathbf{V}'}dS+\int_0^{p_0}\int_S\omega\frac{\partial z'}{\partial p}dSdp+\frac{1}{g}\int_0^{p_0}\int_S\mathbf{V}'\cdot\mathbf{F}'dSdp \qquad (4.20)$$

We find again comparing (4.20) to (3.11) that the part of the integrand depending on the horizontal divergence is missing.

The main difference between an integration of the primitive equations and the quasi-non-divergent equation with respect to the energy conversion from the shear flow to the mean flow is therefore the sign and order of magnitude of the integral

$$-(p_0/g)\int_S \overline{\mathbf{V}}\cdot\overline{(\nabla\cdot\mathbf{V}')\mathbf{V}'}dS$$

Before we try to look into this question we shall specialize the general expressions obtained for the quasi-non-divergent model to the two-parameter case and investigate the ratio between the energy conversion from shear flow to mean flow and the energy conversion between the potential energy and the shear flow. This ratio is a measure of the amount of kinetic energy stored in the shear flow.

5. THE TWO-PARAMETER, QUASI-NON-DIVERGENT CASE

Since the derivations given in the preceding sections separate between the shear flow and the vertical mean flow, we shall in the following use a two-parametric representation of the atmosphere, which makes the same separation. Such a formulation has been given by Eliassen [3] and used by Phillips [7]. Using Phillips' formulation we may write the assumptions in the form

$$\mathbf{V}=\overline{\mathbf{V}}+\mathbf{V}'=\overline{\mathbf{V}}+A(p)\mathbf{V}_T, \qquad \omega=-p_0B(p)\nabla\cdot\mathbf{V}_T \tag{5.1}$$

where $A(p)$ and $B(p)$ are functions satisfying the conditions

$$\overline{A}=0,\ \overline{A^2}=1,\ B(p)=\frac{1}{p_0}\int_0^p A(p)dp \tag{5.2}$$

We may define a vertical velocity ω^* by the relation

$$\frac{2\omega^*}{p_0}=-\nabla\cdot\mathbf{V}_T \tag{5.3}$$

With this notation we can write the prognostic equations in the form:

$$\left.\begin{aligned}&\frac{\partial\overline{\zeta}}{\partial t}+\overline{\mathbf{V}}\cdot\nabla(\overline{\zeta}+f)+\mathbf{V}_T\cdot\nabla\zeta_T=0\\ &\frac{\partial\zeta_T}{\partial t}+\overline{\mathbf{V}}\cdot\nabla\zeta_T+\mathbf{V}_T\cdot\nabla(\overline{\zeta}+f)=\frac{2f_0}{p_0}\,\omega^*\\ &\frac{\partial\psi_T}{\partial t}+\overline{\mathbf{V}}\cdot\nabla\psi_T=\frac{f_0}{p_0}\frac{1}{\lambda^2}\,\omega^*\end{aligned}\right\} \tag{5.4}$$

In (5.4) ψ_T is the stream function for the thermal flow and λ^2 is defined as:

$$\lambda^2=-\frac{f_0^2}{[B^2Tp^{-1}\partial\ln\theta/\partial p]2Rp_0^2} \tag{5.5}$$

The energy conversion from potential energy to kinetic energy of the shear flow is according to (4.20)

$$\{P\cdot K'\}=\int_0^{p_0}\int_S\omega\,\frac{\partial z'}{\partial p}\,dSdp \tag{5.6}$$

which in the two-parameter case reduces to

$$\{P\cdot K'\}=-\frac{2f_0}{g}\int_S\omega^*\psi_T dS \tag{5.7}$$

because

$$\int_0^{p_0}B\,\frac{dA}{dp}\,dp=-1 \tag{5.8}$$

The energy conversion from kinetic energy of the shear flow to the mean flow in this model is

$$\{K'\cdot K\}=-\frac{p_0}{g}\int_S\overline{\zeta'\mathbf{V}'}\cdot\nabla\overline{\psi}dS=-\frac{p_0}{g}\int_S\zeta_T(\mathbf{V}_T\cdot\nabla\overline{\psi})dS \tag{5.9}$$

$\{P\cdot K'\}$ has earlier been computed by the author from atmospheric data as well as for simple sinusoidal flow pattern. It is obvious from (5.9) that $\{K'\cdot\overline{K}\}$ is relatively easy to compute from atmospheric data. Equation (5.9) shows, in fact, in the two-parameter case that we will have an energy conversion from the shear flow to the mean flow, if ζ_T and $(-\mathbf{V}_T\cdot\nabla\overline{\psi})$ are positively correlated. Since $(-\mathbf{V}_T\cdot\nabla\overline{\psi})$ is negative in regions of warm air advection and positive in regions of cold air advection, it is seen, that in order to transfer the kinetic energy from the shear flow to the mean flow we must, on the average, have the cold air advection in regions of cyclonic, relative, thermal vorticity and warm air advection in regions of anticyclonic, relative thermal vorticity. This arrangement of the thermal pattern relative to the mean flow is possible, if the thermal waves on the average are lagging behind the waves in the mean flow. The mean flow loses kinetic energy continuously due to the frictional loss (last integral in (3.10)). In order to maintain the mean flow it is therefore necessary that $\{K'\cdot\overline{K}\}$ is positive and therefore that the thermal waves lag behind the waves in the mean flow. The last result has been obtained by Fjørtoft [4] from somewhat different considerations.

In order to estimate $\{P\cdot K'\}$ and $\{K'\cdot\overline{K}\}$ for simple flow patterns we need an evaluation of the vertical velocity. The equation for the vertical velocity can be obtained from the second and third equations of (5.4) giving:

$$\nabla^2\omega^*-2\lambda^2\omega^*=\frac{p_0}{f_0}\lambda^2[\nabla^2(\overline{\mathbf{V}}\cdot\nabla\psi_T)-\overline{\mathbf{V}}\cdot\nabla\zeta_T-\mathbf{V}_T\cdot\nabla(\overline{\zeta}+f)] \tag{5.10}$$

If we select flow pattern described by

$$\left.\begin{aligned}&\overline{\psi}=-\overline{U}y+\overline{A}\sin(kx)\cos(\mu y),\ k=\frac{2\pi}{L},\ \mu=\frac{\pi}{2W}\\ &\psi_T=-U_Ty+A_T\sin(kx+\alpha_T)\cos(\mu y)\end{aligned}\right\} \tag{5.11}$$

and consider a rectangular region of length L and width $2W$ ($y=0$ in the middle of the channel), it turns out that the solution to (5.10) can be written:

$$\omega^*=R\cos kx\cos\mu y+S\sin kx\cos\mu y+T\sin 2\mu y \tag{5.12}$$

where

$$\left.\begin{aligned} R&=-\frac{p_0}{f_0}\cdot\frac{2k(k^2+\mu^2)U_T\overline{A}-\beta kA_T\cos\alpha_T}{k^2+\mu^2+2\lambda^2}\lambda^2\\ S&=-\frac{p_0}{f_0}\cdot\frac{\beta kA_T\sin\alpha_T}{k^2+\mu^2+2\lambda^2}\lambda^2\\ T&=-\frac{p_0}{f_0}\cdot\frac{2k\mu^3\overline{A}A_T\sin\alpha_T}{4\mu^2+2\lambda^2}\lambda^2\end{aligned}\right\}\tag{5.13}$$

Using the expression (5.12) we can substitute into (5.7) and obtain:

$$\{P\cdot K'\}=LW\frac{p_0}{g}2\lambda^2k\cdot\left[\frac{1}{1+\frac{2\lambda^2}{k^2+\mu^2}}-\frac{1}{2+\frac{\lambda^2}{\mu^2}}\right]\overline{A}A_TU_T\sin\alpha_T\tag{5.14}$$

From the flow pattern (5.11) we can substitute into (5.9), which turns out to be:

$$\{K'\cdot\overline{K}\}=LW\frac{p_0}{g}\frac{1}{2}k(k^2+\mu^2)\overline{A}A_TU_T\sin\alpha_T\tag{5.15}$$

The last term in the bracket in (5.14) is a measure of the energy conversion between the mean potential and mean kinetic energy, which can be seen by separating the fields into a zonal mean and deviations from the zonal mean. Denoting the zonal mean by a subscript Z and the eddies by a subscript E we have:

$$\{P\cdot K'\}=\{P_Z\cdot K'_Z\}+\{P_E\cdot K'_E\}\tag{5.16}$$

and it is easily seen that

$$\{P_Z\cdot K'_Z\}=-LW\frac{p_0}{g}2\lambda^2k\frac{1}{2+\frac{\lambda^2}{k^2}}\overline{A}A_TU_T\sin\alpha_T\tag{5.17}$$

Restricting ourselves to the eddies we have:

$$\{P_E\cdot K'_E\}=LW\frac{p_0}{g}2\lambda^2k\cdot\frac{1}{1+\frac{2\lambda^2}{k^2+\mu^2}}\overline{A}A_TU_T\sin\alpha_T\tag{5.18}$$

Comparing (5.15) and (5.18) it is seen that both of these quantities $\{P_E\cdot K'_E\}$ and $\{K'\cdot\overline{K}\}$, are positive if $\alpha_T>0$, i.e., if the temperature field is lagging behind the pressure field. It is further seen that the two conversions depend on the amplitudes and the thermal zonal wind in the same way, but that they depend differently on the scale of the motion.

The dependence on the scale of the motion for $\{P_E\cdot K'_E\}$ was computed earlier by the author (Wiin-Nielsen [10]). It suffices therefore here to consider the ratio between $\{K'\cdot\overline{K}\}$ and $\{P_E\cdot K'_E\}$. We get

$$\{K'\cdot\overline{K}\}/\{P_E\cdot K'_E\}=\frac{1}{2}+\frac{k^2+\mu^2}{4\lambda^2}\tag{5.19}$$

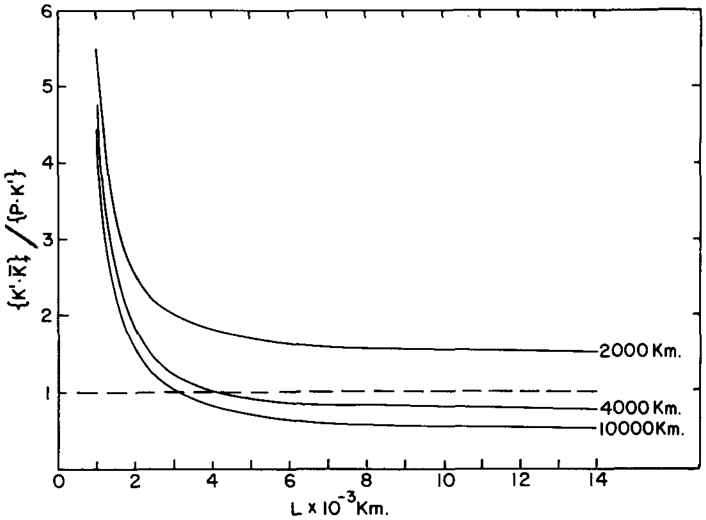

FIGURE 1.—The ratio of energy conversion between shear flow and mean flow to the conversion between available potential energy and shear flow kinetic energy as a function of zonal wavelength for different values of the meridional wavelength.

The ratio (5.19) is illustrated in figure 1 for different values of the meridional scale as a function of the zonal scale with $\lambda^2=2.5\times10^{-12}$ m.$^{-2}$ (2W) is actually a half wavelength in the meridional direction.

The essential results illustrated in figure 1 are that for the small-scale motion a larger amount of kinetic energy is transformed into kinetic energy of the mean flow than is converted from the available potential energy. On the small scale there is therefore a depletion of the kinetic energy in the shear flow. On the other hand, the figure shows also that an accumulation of kinetic energy takes place on the large scale in the shear flow, since the amount transformed to kinetic energy of the mean flow is smaller than the amount converted from potential energy as long as the meridional scale is large. On the medium scale (L=3,000–5,000 km.) there is no storage of kinetic energy in the shear flow, again as long as the meridional scale is large enough. Since $\{P_E\cdot K'_E\}$ has a maximum around 3,000–5,000 km., where the baroclinic instability is largest (Wiin-Nielsen [10]), our results show that no storage takes place in the greatest-amplitude waves in the shear flow.

As is seen from the figure and from (5.19) the ratio becomes very large, if the scale is small (μ and k large). Suppose now that we have some positive or negative conversion from potential to kinetic energy on the small scale. According to (5.19) we should therefore expect a rather violent reaction in the conversion between shear flow and mean flow kinetic energy. Since the flow at a certain pressure level (usually 600 or 500 mb.) is used to represent the mean flow, the latter fact may explain why we often find appreciable small-scale noise in the predictions with baroclinic models. This noise is not solely due to the numerical procedures, but is aggravated by the physical properties of the quasi-non-divergent model.

6. CALCULATION OF ENERGY CONVERSION FROM OBSERVED DATA

The energy conversion between shear flow and mean flow kinetic energies can be evaluated from observed data as remarked at the end of section 3. The computations were made with data from January 1959, using 850 and 500-mb. data. These data were selected because they entered into an earlier computation of energy conversion between available potential and kinetic energy (Wiin-Nielsen [10]). Both of the integrals in (3.12) were approximated and evaluated once a day using a procedure as described below.

With data available only at two levels for January 1959 we are forced to use a two-parameter representation of the type given in section 5. As shown in that section we may write the basic expression (3.14) in the form

$$\{K'\cdot\overline{K}\}=-\frac{p_0}{g}\int_S[\nabla\cdot\mathbf{V}_T(\overline{\mathbf{V}}\cdot\mathbf{V}_T)+\zeta_T(\overline{\mathbf{V}}\times\mathbf{k})\cdot\mathbf{V}_T]dS \quad (6.1)$$

We introduce the notations:

$$\left.\begin{aligned}\{K'\cdot\overline{K}\}_{ND}&=-\frac{p_0}{g}\int_S\zeta_T(\overline{\mathbf{V}}\times\mathbf{k})\cdot\mathbf{V}_T dS\\ \{K'\cdot\overline{K}\}_{D}&=-\frac{p_0}{g}\int_S\nabla\cdot\mathbf{V}_T(\overline{\mathbf{V}}\cdot\mathbf{V}_T)dS\end{aligned}\right\} \quad (6.2)$$

Using the formulation (5.1) it is easily seen that

$$\zeta_T=(A_{50}-A_{85})^{-1}\zeta_d \quad (6.3)$$

where ζ_d is the thermal vorticity in the layer between 850 and 500 mb. and A_{50} and A_{85} are representative values of $A(p)$ at the levels 50 and 85 cb. A_{50} and A_{85} were taken from the table given by Eliassen [3].

We find further that

$$(\overline{\mathbf{V}}\times\mathbf{k})\cdot\mathbf{V}_T=(A_{50}-A_{85})^{-1}\mathbf{J}(\psi_d,\psi_{50}) \quad (6.4)$$

and therefore that

$$\{K'\cdot\overline{K}\}_{ND}=\frac{p_0}{g(A_{50}-A_{85})^2}\int_S\zeta_d\mathbf{J}(\psi_{50},\psi_d)\,dS \quad (6.5)$$

The integral (6.5) was evaluated by computing the thermal, relative vorticity and the Jacobian at the grid points in the JNWP octagonal grid. These values were next interpolated to a latitude-longitude grid using a grid size of 2.5°. The integral may then conveniently be evaluated using the form:

$$\{K'\cdot\overline{K}\}_{ND}=\frac{p_0a^2}{g(A_{50}-A_{85})^2}\int_{\phi_0}^{\frac{\pi}{2}}L(\phi)\cos\phi d\phi \quad (6.6)$$

where $L(\phi)$ is defined by the equation

$$L(\phi)=\int_0^{2\pi}\zeta_d\mathbf{J}(\psi_{50},\psi_d)d\lambda \quad (6.7)$$

a is the radius of the earth, ϕ is latitude, and λ longitude in the preceding formulas. The southernmost latitude, ϕ_0, was taken to be 20° N. Both of the integrals in (6.6) and (6.7) were evaluated using finite sums and an increment of 2.5°.

The second integral in (6.2) may be evaluated in a similar way. A special problem arises due to the presence of the divergence. This quantity was evaluated using the available vertical velocities which are supposed to apply at the 600-mb. level according to the model assumptions in the JNWP operational model. We get, using (5.1), that

$$\nabla\cdot\mathbf{V}_T=-\frac{\omega_{60}}{p_0B_{60}}\simeq-\frac{\omega_{60}}{42} \quad (6.8)$$

again using Eliassen's [3] estimates of $B(p)$.

$$\{K'\cdot\overline{K}\}_D=\frac{1}{gB_{60}(A_{50}-A_{85})}\int_S\omega_{60}\cdot\left(\mathbf{V}_{50}\cdot\mathbf{V}_\alpha-\frac{A_{50}}{A_{50}-A_{85}}\mathbf{V}_\alpha^2\right)dS \quad (6.9)$$

or

$$\{K'\cdot\overline{K}\}_D=\frac{a^2}{gB_{60}(A_{50}-A_{85})}\int_{\phi_0}^{\frac{\pi}{2}}I(\phi)\cos\phi d\phi \quad (6.10)$$

where

$$I(\phi)=\int_0^{2\pi}\omega_{60}P(\psi_5,\psi_a)d\lambda \quad (6.11)$$

and

$$P(\psi_5,\psi_d)=\nabla\psi_5\cdot\nabla\psi_d-\frac{A_{50}}{A_{50}-A_{85}}(\nabla\psi_d)^2 \quad (6.12)$$

The integrals (6.10) and (6.11) were again evaluated by finite sums using a 2.5° grid size.

The mean values for the month of January 1959 from 31 evaluations for $\{K'\cdot\overline{K}\}_{ND}$ turned out to be 4.3×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$ while $\{K'\cdot\overline{K}\}_D$ was -0.47×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$. We find therefore that $\{K'\cdot\overline{K}\}_D$ is slightly more than 10 percent of the values for $\{K'\cdot\overline{K}\}_{ND}$. The standard deviations of the two mean values are 1.5×10^{-4}kj. m.$^{-2}$ sec.$^{-1}$ and 1.1×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$.

It should further be mentioned that the total energy conversion

$$\{K'\cdot\overline{K}\}_{ND}+\{K'\cdot\overline{K}\}_D$$

turned out to be positive for each day, which means that the transformation constantly goes from the shear flow kinetic energy to the mean flow kinetic energy.

Some remarks should be made at this point regarding the approximations which are used in evaluating the two energy transformation integrals. The first integral is evaluated using non-divergent winds at both levels. The balance equation was solved for the stream function at the 500 and 850-mb. levels and ψ_d was obtained by subtraction. This integral is therefore evaluated as it would be in a quasi-non-divergent model. The second integral is also evaluated using vertical velocities and non-divergent winds from an adiabatic, frictionless, and quasi-non-divergent model. Such an evaluation is naturally an approximation because the integral is connected with the advection with divergent wind components. However,

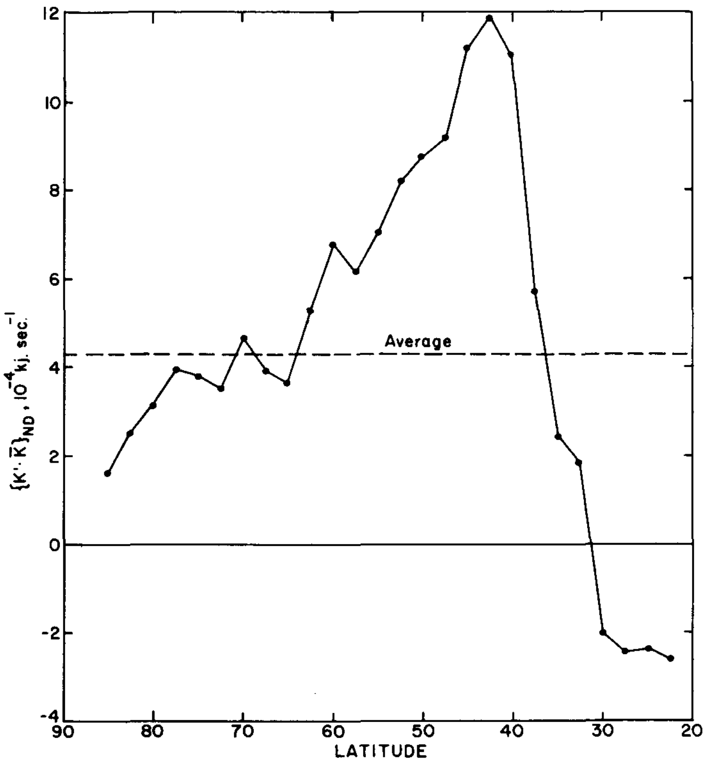

FIGURE 2.—The correlation between thermal relative vorticity and temperature advection expressed as an energy conversion as a function of latitude.

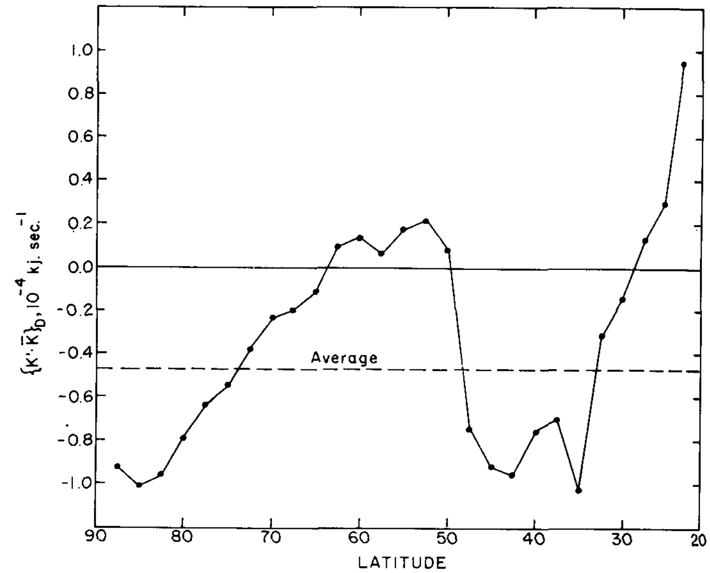

FIGURE 3.—Contributions to the total conversion of shear flow kinetic energy to mean flow kinetic energy from the divergent part of the flow from different latitude bands.

diagnostic computations of the divergent wind components have shown that they are small compared to the non-divergent components, and the evaluation is therefore a good first approximation.

The two averaged figures for the conversion from shear flow to mean flow kinetic energy may be compared with the conversion from available potential to shear flow kinetic energy. The latter conversion was computed earlier by the author (Wiin-Nielsen [10]), using data from the same month, to be 14.0×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$. The total conversion from shear flow to mean flow is 3.8×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$, which means according to these estimates that about 27 percent of the available potential energy, which is converted, eventually gets into the mean flow kinetic energy, where it is dissipated through friction. The dissipation is measured by the last integral in (3.12). However if a quasi-non-divergent model is used about 30 percent of the converted available potential energy goes into the mean flow kinetic energy. These numbers suggest that a difference of about 10 percent will exist between forecasts made with the most simple quasi-non-divergent model and a more advanced type of prediction model based upon the vorticity equation or on the equations of motion themselves.

The ratio between the two energy conversions, $\{K'\cdot\bar{K}\}$ and $\{P\cdot K'\}$, is measured to be somewhat smaller than the estimate obtained from (5.19) which was evaluated using a linearized approach. The difference between the two results could indicate that there is a systematic underestimate of $\{K'\cdot\bar{K}\}$. If this is the case, it is probably due to the fact that only data from the lower part of the troposphere have been used in the observational part of the study. From a similar study using a greater vertical resolution one would be able to tell whether or not the suggested explanation is correct.

Figure 2 illustrates the contribution from the different latitude bands to the integral $\{K'\cdot\bar{K}\}_{ND}$. The ordinate is given in the units of an energy conversion, but due to the fact that the latitude rings cannot be considered as energetically closed with any degree of approximation one should think of the curve in figure 2 as illustrating the correlation between the thermal, relative vorticity and the temperature advection. It is seen that the result of the observational study agrees with the remarks given in section 5 in the sense that cold (warm) air advection is positively correlated with regions of cyclonic (anticyclonic), thermal vorticity, or in other words that the temperature field on the average lags behind the height field. An exception to this is found in the low-latitude part of the region south of 30° N. where a negative correlation exists. It is further seen that the greatest contribution is found in the middle latitudes with the maximum appearing at 42.5° N.

Figure 3 contains the curve illustrating the contributions from the latitude bands to the integral $\{K'\cdot\bar{K}\}_{D}$. We find a positive maximum at 52.5° N. and minima at 85° N., 42.5° N., and 35° N. A comparison between figure 2 and figure 3 shows that $\{K'\cdot\bar{K}\}_{D}$ at all places is small compared to $\{K'\cdot\bar{K}\}_{ND}$. One may therefore state that the major part of the energy conversion between shear flow and mean flow is contained in the quasi-non-divergent model.

7. ENERGY CONVERSIONS IN THE WAVE NUMBER REGIME

As in earlier studies of energy conversions computed from observed atmospheric data, it has been found of interest to compute the energy conversion between the vertical shear flow and mean flow in the wave number regime (Wiin-Nielsen [10], Saltzman and Fleischer [9]). The technique which has been used in this study is slightly different from the one used in the earlier study by the author. The main steps in the computations will be described below for the case of the conversion $\{K'\cdot\overline{K}\}_{ND}$ in the two-parameter case.

$\{K'\cdot\overline{K}\}_{ND}$ is in this case given by the following formula (see (6.6))

$$\{K'\cdot\overline{K}\}_{ND}=\frac{p_0a^2}{g(A_{50}-A_{85})^2}\int_{\phi_0}^{\frac{\pi}{2}}L(\phi)\cos\phi d\phi \qquad (7.1)$$

where the notations have been explained in section 6.

Assuming next that the thermal relative vorticity and the temperature advections are written in the form

$$\zeta_d=a_0(\phi)+\sum_{n=1}^{N}\{a_n(\phi)\sin(n\lambda)+b_n(\phi)\cos(n\lambda)\} \qquad (7.2)$$

and

$$J(\psi_{50},\psi_d)=A_0(\phi)+\sum_{n=1}^{N}\{A_n(\phi)\sin(n\lambda)+B_n(\phi)\cos(n\lambda)\} \qquad (7.3)$$

where

$$\left.\begin{aligned} a_0(\phi)&=\frac{1}{\pi}\int_0^{2\pi}\zeta_d d\lambda\\ a_n(\phi)&=\frac{1}{2\pi}\int_0^{2\pi}\zeta_d\sin(n\lambda)d\lambda\\ b_n(\phi)&=\frac{1}{2\pi}\int_0^{2\pi}\zeta_d\cos(n\lambda)d\lambda \end{aligned}\right\} \qquad (7.4)$$

and corresponding expressions apply for $A_0(\phi)$, $A_n(\phi)$, and $B_n(\phi)$, we can write $L(\phi)$ in the following form:

$$L(\phi)=L_0(\phi)+\sum_{n=1}^{N}L_n(\phi) \qquad (7.5)$$

where

$$L_0(\phi)=2\pi a_0(\phi)A_0(\phi) \qquad (7.6)$$

and

$$L_n(\phi)=\pi[a_n(\phi)A_n(\phi)+b_n(\phi)B_n(\phi)] \qquad (7.7)$$

We may then finally write $\{K'\cdot\overline{K}\}_{ND}$ in the following form

$$\{K'\cdot\overline{K}\}_{ND}=\{K'\cdot\overline{K}\}_{ND}^{(0)}+\sum_{n=1}^{N}\{K'\cdot\overline{K}\}_{ND}^{(n)} \qquad (7.8)$$

where

$$\{K'\cdot\overline{K}\}_{ND}^{(n)}=\frac{p_0a^2}{g(A_{50}-A_{85})^2}\int_{\phi_0}^{\frac{\pi}{2}}L_n(\phi)\cos\phi d\phi \qquad (7.9)$$

The values of the thermal, relative vorticity and the temperature advection were as before computed in all interior points of the JNWP octagonal grid. Values were then obtained in a latitude-longitude grid with a grid size of 2.5° by interpolation. The Fourier coefficients $a_0(\phi)$, $a_n(\phi)$, $b_n(\phi)$, $A_0(\phi)$, $A_n(\phi)$, and $B_n(\phi)$ were then computed for each latitude circle from 20° N. to 87.5° N. (a total of 28 values of each coefficient) using finite sums in the evaluation of the integrals in (7.4) and the corresponding formulas for $A_0(\phi)$, $A_n(\phi)$, and $B_n(\phi)$. The final values of $\{K'\cdot\overline{K}\}_{ND}^{(n)}$ were then computed from (7.9) replacing the integral by a finite sum.

In the earlier evaluation of the conversion between available potential energy and kinetic energy (Wiin-Nielsen [10]) it was found that the waves with wave numbers larger than 10 gave very small contributions to the total spectrum. It was therefore decided to set $N=10$.*

A completely analagous procedure was used to evaluate the integral $\{K'\cdot\overline{K}\}_D$ in (6.10). The Fourier analysis was here performed on the quantities ω_{60} and P (ψ_5, ψ_d), and N was again equal to 10.

The calculations were carried out for each day of January 1959. The results consist of the 11 values of $\{K'\cdot\overline{K}\}_{ND}^{(n)}$ for $0\leq n\leq 10$ and corresponding values of $\{K'\cdot\overline{K}\}_{D}^{(n)}$ for the same values of n. The sums $\sum_{n=0}^{10}\{K'\cdot\overline{K}\}_{ND}^{(n)}$ and $\sum_{n=0}^{10}\{K'\cdot\overline{K}\}_{D}^{(n)}$ can be compared with the total values of $\{K'\cdot\overline{K}\}_{ND}$ and $\{K'\cdot\overline{K}\}_D$ computed in section 6. The comparison gives how large a fraction of the total energy conversion we can explain by the first 10 wave numbers. We find in both cases that the contribution from wave numbers with $n>10$ is not negligible (compare the footnote).

Figure 4 shows the spectrum of $\{K'\cdot\overline{K}\}_{ND}$ computed as the average over the spectra for the individual days of January 1959. The unit is 10^8 kj. sec.$^{-1}$, and the figure is comparable with the corresponding figure for the conversion from available potential energy to kinetic energy in Wiin-Nielsen [10]. The figure shows a maximum conversion of kinetic energy from the shear flow to the mean flow for $n=7$, which is almost the same scale on which we find the maximum conversion between potential and kinetic energy of the shear flow. There is an indication of a second maximum on the planetary scale ($n=1$), but this maximum is not as pronounced as the corresponding maximum in $\{P\cdot K'\}$. We arrive therefore at the tentative conclusion that the baroclinic waves with an amplification rate close to the maximum rate are the most important in the maintenance of the kinetic energy of the vertical mean flow against frictional dissipation.

Table 1 gives the mean values and standard deviations (S) of the energy conversions $\{K'\cdot\overline{K}\}_{ND}$ corresponding to figure 4. The rather large values of the standard deviations presented in table 1 show that there is a considerable scatter around the mean values in figure 4 and table 1.

In view of the theoretical results derived in section 5 from a simple linear treatment, it is interesting to compute

*In retrospect it turns out that it would have been better to use a somewhat larger value of N, because the thermal relative vorticity and the temperature advection both tend to have considerable amplitudes for large values of the wave number.

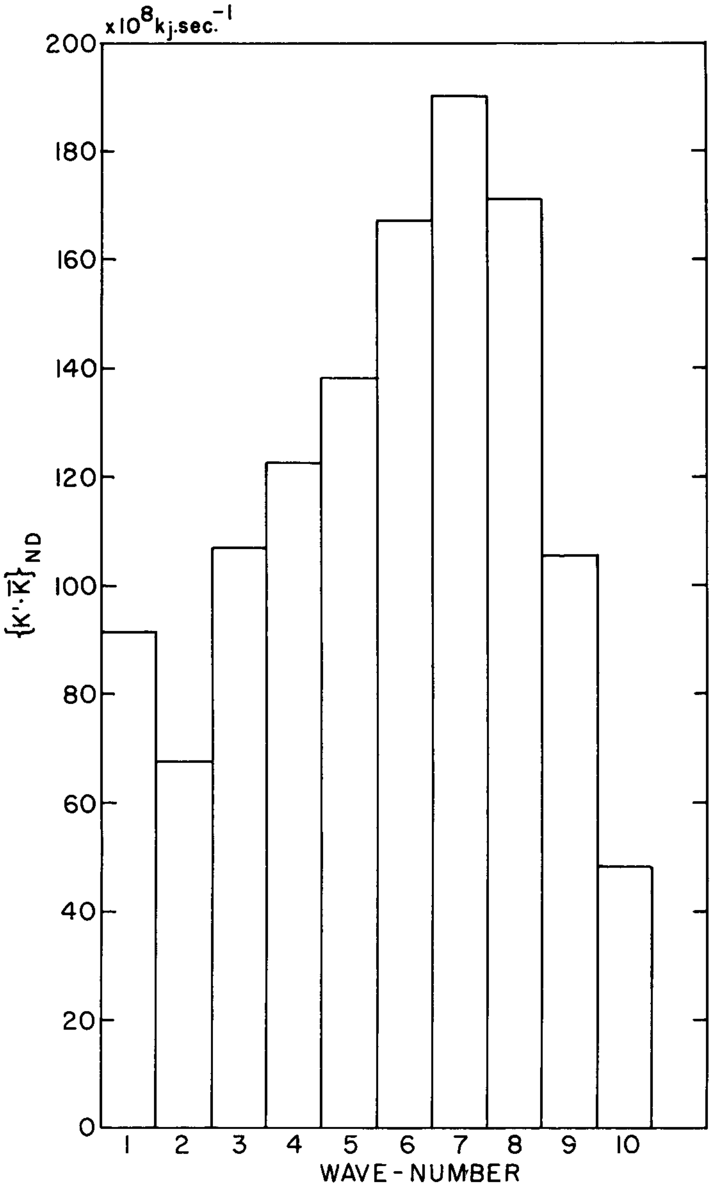

FIGURE 4.—The energy conversion $\{K'\cdot\overline{K}\}_{ND}$ as a function of wave number averaged in time, for January 1959. The horizontal coordinate is number of waves around the hemisphere, while the vertical coordinate is energy conversion per unit time north of about 20° N.

the ratio between the energy conversion between the kinetic energies of the shear flow and the mean flow and the conversion between potential energy and the kinetic energy of the shear flow. This ratio is reproduced in table 2. The latter energy conversion has been taken from Wiin-Nielsen [10].

The figures in table 2 show naturally a more irregular variation than the curves in figure 1, but there is a qualitative agreement to the extent that the ratio between the two energy conversions tends to increase for large values of the wave number in the theoretical curves as well as in the figures obtained from observations.

Figure 5 shows the distribution of $\{K'\cdot\overline{K}\}_D$ as a function of wave number in the average for January 1959. We notice first of all that the magnitude of $\{K'\cdot\overline{K}\}_D$ is small compared to $\{K'\cdot\overline{K}\}_{ND}$ for all wave numbers. This result agrees with the result obtained in section 6, where we considered the total conversion over all wave numbers. We find further agreement to the extent that $\{K'\cdot\overline{K}\}_D$ is negative for most wave numbers. The only exception is the small positive conversions for wave numbers 5 and 6 given in figure 5.

Table 3 gives the mean values and standard deviations (S) corresponding to figure 5. We find as before that the standard deviations show a large scatter of the individual daily values around the monthly mean value.

Table 4 contains the values of the ratio$|\{K'\cdot\overline{K}\}_D/\{K'\cdot\overline{K}\}_{ND}|$ as a function of wave number. The numbers in this table are obtained from the average values given in tables 1 and 3. Table 4 shows the ratio is quite small for $n\geq 4$. If these results are significant, we arrive at the conclusion that the major part of the energy conversion between the kinetic energies of the shear flow and the vertically averaged flow is contained in $\{K'\cdot\overline{K}\}_{ND}$. Some reservation must be taken to this conclusion due to the fact that we have used divergences computed from a quasi-non-divergent model to evaluate $\{K'\cdot\overline{K}\}_D$. If the vertical velocities and therefore also the divergence implied by such a model are systematically too small, it is evident that the conclusion above could be radically changed. Some insight into this question can be gained by evaluating the two conversions $\{K'\cdot\overline{K}\}_{ND}$ and $\{K'\cdot\overline{K}\}_D$ using data

TABLE 1.—*Mean values and standard deviations* (S) *of the energy conversion* $\{K'\cdot\overline{K}\}_{ND}$ *as a function of wave number* n, *corresponding to figure 4*

n	0	1	2	3	4	5	6	7	8	9	10
$\{K'\cdot\overline{K}\}_{ND}\times 10^{-8}$	77.8	91.6	67.4	107.1	122.7	138.2	167.6	190.1	171.1	105.4	48.4
$S\times 10^{-8}$	70.9	105.4	91.6	110.6	91.6	157.3	157.3	179.7	174.5	84.7	81.2

TABLE 2.—*Ratio of the energy conversion* $\{K'\cdot\overline{K}\}_{ND}$ *to the energy conversion* $\{P\cdot K'\}$ *as a function of wave number* n

n	0	1	2	3	4	5	6	7	8	9	10
$\{K'\cdot\overline{K}\}_{ND}/\{P\cdot K'\}$	0.38	0.93	0.25	0.40	0.54	0.63	0.50	0.59	0.83	0.78	1.38

from extended numerical integrations of models based upon the primitive equations.

For the planetary waves ($1 \leq n \leq 3$) we find somewhat larger values of the ratio given in table 4. This would indicate that $\{K' \cdot \overline{K}\}_D$ is more important for these waves than it is for the smaller scales in describing the conversion of kinetic energy between the shear flow and the mean flow. We are again forced to express some reservations to this conclusion. It has earlier been pointed out (Wiin-Nielsen [10]) that the vertical velocities implied by a quasi-non-divergent two-parameter model could be radically changed by heat sources and sinks. Since the evaluation of $\{K' \cdot \overline{K}\}_D$ in this paper makes use of vertical velocities computed from an adiabatic, frictionless model, it is evident that the discussion given in the earlier paper [10] also applies here.

With the reservations mentioned above we conclude therefore that the major part of the energy conversion $\{K' \cdot \overline{K}\}$ is contained in $\{K' \cdot \overline{K}\}_{ND}$.

8. ESTIMATES OF DECAY TIMES

In connection with the computations we have made it is also of interest to compute some measure of the total amounts of kinetic energy in the vertically averaged flow and in the shear flow. A crude estimate can be made of the ratio of these quantities from the model assumptions in a two-parameter model. Suppose that the wind variations with pressure were given by (5.1) with the function $A(p)$ satisfying (5.2). We would then have

$$\overline{K} = \frac{1}{g} \int_0^{p_0} \int_S \frac{1}{2} \overline{\mathbf{V}}^2 dS dp = \frac{p_0}{2g} S \tilde{\overline{\mathbf{V}}}^2 \tag{8.1}$$

where the tilde ($\sim$) means an area average.

The kinetic energy in the shear flow would be:

$$K' = \frac{1}{g} \int_0^{p_0} \int_S \frac{1}{2} \mathbf{V}'^2 dS dp = \frac{1}{g} \int_0^{p_0} \int_S \frac{1}{2} A(p)^2 \mathbf{V}_T^2 dS dp = \frac{p_0}{2g} S \tilde{\mathbf{V}}_T^2 \tag{8.2}$$

A crude first estimate of $\overline{K}/K'$ can be obtained from empirical data (Wiin-Neilsen [12]), which show that $|\overline{\mathbf{V}}| \sim 1.8 |\mathbf{V}_T|$, which means that the mean flow kinetic energy is about three times larger than the shear flow kinetic energy. While the kinetic energy of the vertically averaged flow thus is a few times larger than the kinetic energy in the shear flow it receives only a small fraction of the total amount by conversion from shear flow energy and loses naturally in the average the same amount by frictional dissipation. The amount of shear flow kinetic energy, on the other hand, is a few times smaller, but a larger fraction is received by conversion from available potential energy and the same amount is naturally on the average lost by conversion to mean flow kinetic energy and by frictional dissipation.

We may estimate the total decay time* in the two energy reservoirs of mean flow and shear flow kinetic energy. In order to do this we need an estimate of the

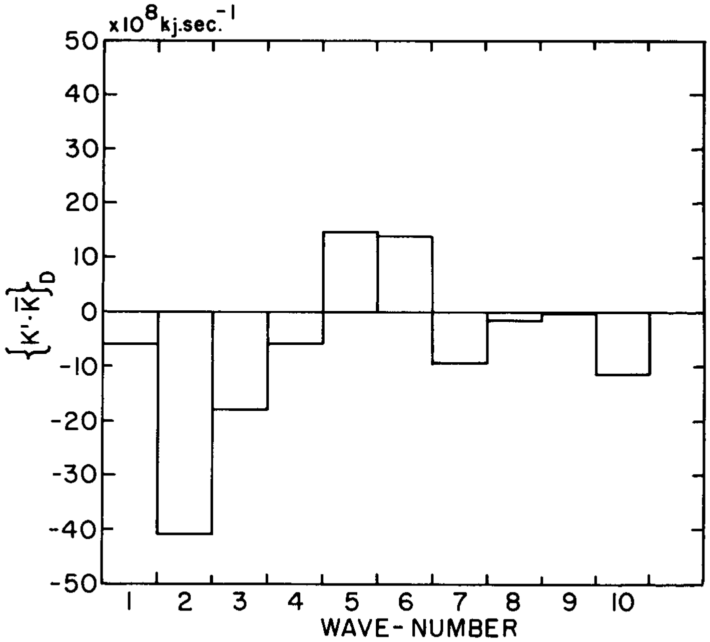

Figure 5.—The energy conversion $\{K' \cdot \overline{K}\}_D$ as a function of wave number averaged in time, for January 1959.

*The total decay time is the time it would take to empty an energy reservoir completely, if the energy supplies were cut off.

Table 3.—*Mean values and standard deviation* (S) *of the energy conversion* $\{K' \cdot \overline{K}\}_D$ *as a function of wave number* n, *corresponding to figure 5*

n	0	1	2	3	4	5	6	7	8	9	10
$\{K' \cdot \overline{K}\}_D \times 10^{-8}$	−45.5	−6.0	−41.1	−18.0	−5.9	14.8	14.1	−9.4	−1.6	−0.3	−11.8
$S \times 10^{-8}$	32.1	30.1	35.8	43.5	29.0	39.4	37.0	37.8	31.6	22.6	25.1

Table 4.—*Ratio of the energy conversion* $\{K' \cdot \overline{K}\}_D^{(n)}$ *to the energy conversion* $\{K' \cdot \overline{K}\}_{ND}^{(n)}$ *as a function of wave number* n

n	0	1	2	3	4	5	6	7	8	9	10
$\lvert \{K' \cdot \overline{K}\}_D^{(n)} / \{K' \cdot \overline{K}\}_{ND}^{(n)} \rvert$	0.58	0.07	0.61	0.17	0.05	0.11	0.08	0.05	0.01	0.00	0.24

amount of energy in the two reservoirs. As a first approximation we have taken the total amount of kinetic energy in the atmosphere in the wintertime as estimated by Pisharoty [8] and reproduced by Bjerknes [1]. In the layer between sea level and 10 cb. it amounts to 38.4×10^{16} kj. for the latitude band 17° to 77° N. or 22.6×10^{2} kj. m.$^{-2}$. This amount was divided between the mean flow and shear flow kinetic energies in the ratio 3:1, which gives $\overline{K}=16.95\times10^{2}$ kj. m.$^{-2}$ and $K'=5.65\times10^{2}$ kj. m.$^{-2}$. The total energy influx into the reservoir of mean flow kinetic energy is as estimated in this paper 3.8×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$, which gives a total decay time of 52 days. If we only consider the contribution from the quasi-non-divergent part we get an influx of 4.3×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$, which gives the total decay time of 46 days.

The total energy influx into the reservoir of shear flow kinetic energy from conversion from available potential energy is 14.0×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$ (Wiin-Nielsen [10]) but of this amount 3.8×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$, goes into the mean flow kinetic energy, which leaves 10.2×10^{-4} kj. m.$^{-2}$ sec.$^{-1}$ in the shear flow kinetic energy. With $K'=5.65\times10^{2}$ kj. m.$^{-2}$ we get a total decay time of 6.4 days, which then also is an estimate of the frictional dissipation measured by the last term in (3.11).

9. SUMMARY AND CONCLUSIONS

The total kinetic energy of the horizontal, hydrostatic flow in the atmosphere may be divided into the kinetic energy of the vertically integrated flow and the kinetic energy of the deviation from this flow, the so-called shear flow. It is shown that the kinetic energy gained by conversion from potential energy goes into the kinetic energy of the shear flow. The energy transformation between shear flow and mean flow is found in general and also in the special case of the quasi-non-divergent model. The general formula for energy transformation between shear flow and mean flow may be shown to consist of two terms of which one is formally represented in the quasi-non-divergent model, while the other will be present in more advanced models based upon the vorticity equation or the primitive equations.

The energy conversion between shear flow and mean flow and between available potential energy and shear flow kinetic energy is evaluated in the quasi-non-divergent case using a two-parameter representation of the atmosphere. It is especially shown that the ratio between the two energy conversions tends to become large for small-scale motion, but less than unity for planetary flow. On the intermediate Rossby scale we find a ratio close to unity which means that no storage takes place in the shear flow kinetic energy on this scale. These results are obtained using linear equations with finite amplitude disturbances superimposed on a zonal current which varies only with pressure.

Observed data have been used to evaluate the energy conversion between shear flow and mean flow in a quasi-non-divergent two-parameter model. Data from such a model have also been used to estimate the conversion due to the divergent part of the flow. It is found that the latter is only about 10 percent of the former indicating that the largest part of the energy conversion in question is present in a quasi-non-divergent model, and that only a small part will be added in more advanced models, especially models based upon the primitive equations.

It is further found that only about 27 percent (in quasi-non-divergent models 30 percent) of the energy converted from available potential energy goes into the kinetic energy of the vertically averaged flow.

The energy conversion $\{K'\cdot\overline{K}\}$ has also been evaluated as a function of wave number. It is found that $\{K'\cdot\overline{K}\}_{ND}$ is positive on the average and has a maximum around wave number 7. The conversion $\{K'\cdot\overline{K}\}_{D}$ is numerically much smaller and tends to be negative for most wave numbers.

Wiin-Nielsen and Brown [11] have recently estimated the generation of available potential energy from exactly the same data as have been used in this study. It turns out that the generation of zonal available potential energy amounts to 50×10^{-4} kj.m.$^{-2}$ sec.$^{-1}$ on the average for January 1959. Since the conversion from available potential energy to shear flow kinetic energy is estimated to be 14.4 kj.m.$^{-2}$ sec.$^{-1}$, we find that 35.6 kj.m.$^{-2}$ sec.$^{-1}$ is being dissipated from the reservoir of potential energy, or in other words 71 percent of the generation of zonal available potential energy. Most of the dissipation is due to a degradation of eddy available potential energy by diabatic processes. The conversion from shear flow kinetic energy (3.8 kj.m.$^{-2}$ sec.$^{-1}$) found in this paper means that 10.6 kj.m.$^{-2}$ sec.$^{-1}$ or 21 percent of the generation of zonal available potential energy is dissipated from the reservoir of shear flow kinetic energy. As a residual we find that 8 percent is dissipated from the reservoir of mean flow kinetic energy.

The most surprising result is probably the small fraction dissipated from the mean flow kinetic energy. If our estimate of $\{K'\cdot\overline{K}\}$ as suggested earlier is too small we would find a greater fraction being dissipated from the mean flow kinetic energy. On the other hand, if the result is correct it shows that the total decay time of the mean flow kinetic energy is very large.

ACKNOWLEDGMENTS

The author would like to thank Dr. J. Smagorinsky, who originally directed his attention toward the advantages of the special partition of the kinetic energy made in this paper. Further thanks are due to Lt. Col. H. Ellsaesser for permission to use his interpolation and Fourier analysis code, to Mrs. Lena Loman and Mrs. J. McClees for coding all the numerical calculations contained in this paper, and to Mrs. Charmian Jeffries, who has performed the processing of the computed results.

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